1 Infinite points vs. finite points

(First draft)

In dealing with the so-called *infinite points* in the framework of 3DE the question arises, under which circumstances a point may be declared to be infinite. In fact, the concept of infinity is an idealized one and is not realized on earth (and maybe not even in the universe).

Declaring a point to be *infinite* means to declare all rays going from the camera positions of a sequence to that point to be parallel. Infinite points cannot be used to determine the position of a camera, however, they can be used to determine the camera orientation. Since in reality no point can be at *infinity* we should specify, how far a point must be away from the camera, so that it can be declared to be infinite without loss of precision of the camera motion path calculations. In practice, there are many situations in which it is intuitively clear, that a point can be declared to be infinite: In case the camera is moving around within a range of a few meters, the moon, clouds, stars, and the horizon can be used for defining infinite points.

First we need to find a criterion that specifies what we mean by saying "the point is too near to be infinite, so that the calculation becomes unprecise" or "the point is far enough to be declared to be infinite". When we declare a point to be infinite, we introduce a certain error (nevertheless we do it because we gain stability in the 3d-calculations): the 2d-position refers to a point which is finite in reality but infinite for 3DE's calculation kernel. This 2d-error has to be compared to the other 2d-tracking errors for a given sequence. So, let us define: a point is too near to be declared to be infinite, when setting it to infinity causes a 2d-error bigger than the common 2d-tracking error of the project.

We will not be able to give a general expression, because the limit distance depends on many factors. However, we can construct a simplified situation, which should at least give us an order of magnitude we can work with. This simplified situation is as follows:

We consider a camera motion math and a point at a large distance as in the accompanying figure. The point lies at a distance of D, the component of the camera path orthogonal to the average distance vector from the camera to the point has a length of L. For simplicity we assume, that the tracking curve of the point lies near the center of the image and has only a horizontal component (i.e. horizontal in 2d-screen coordinates). The ray directions 3DE assumes when the point is declared to be infinite deviate from the actual ray directions in 3d-space. The error is expressed by a time-dependent angle $\alpha(t)$. We denote the camera motion path by $x(t)$. For simplicity, we will also assume, that the component of the camera path in direction to the point is small against D so that we can set it to zero. That means, we use the (freely choosable) parametrization

$$(x(t), 0, 0) = (\frac{L}{T}t, 0, 0) \tag{1}$$

where t runs from $-T/2$ to $+T/2$. Let us calculate the (square) error of the ray direction. From the figure it is clear that

$$\tan(\alpha(t)) = \frac{x(t)}{D} \tag{2}$$

We are dealing with small angles, therefore we can replace $\tan(\alpha(t))$ by α. The square error $\langle\alpha^2\rangle$ resulting from integration along the camera motion path reads

$$\begin{aligned} \langle\alpha^2\rangle &= \frac{1}{T}\int_{-\frac{T}{2}\leq t\leq +\frac{T}{2}} \left(\frac{x(t)}{D}\right)^2 dt \\ &= \frac{1}{12}\frac{L^2}{D^2} \end{aligned} \tag{3}$$

The question is now, what is this error expressed in pixel? Let us assume, we have a camera with focal length f and filmback width w_{fb}. The images are digitized with a horizontal resolution of w_{fov} pixel. We calculate the dependency between small angle displacements near the center of the image and the resulting displacement expressed in pixel. A point at a distance of σ in pixel from the image center results from projecting a point at an angle of

$$\phi = 2\arctan\frac{\sigma w_{\text{fb}}}{f w_{\text{fov}}} \tag{4}$$

from the direction of view with respect to the camera position. For small ϕ and σ this reads

$$\phi = \frac{2\sigma w_{\text{fb}}}{f w_{\text{fov}}} \tag{5}$$

The error angle we calculated above can now be expressed in pixel:

$$\sigma = \sqrt{\langle\alpha^2\rangle}\frac{f w_{\text{fov}}}{2 w_{\text{fb}}} \tag{6}$$

Using equation (3) we get the relation

$$\sigma = \frac{1}{\sqrt{48}}\frac{L}{D}\frac{f w_{\text{fov}}}{w_{\text{fb}}} \tag{7}$$

2 Example

Assume we are given a sequence digitized with a horizontal resolution of 1000 pixel. The camera may have a filmback width of 40mm and a focal length of 50mm. The camera moves about 2m and the point being considered is at a distance of 100m. What is the average pixel error we make by declaring the point to be infinite?

$$\sigma = \frac{1}{\sqrt{48}}\frac{2\text{m}}{100\text{m}}\frac{50\text{mm}1000\text{pixel}}{40\text{mm}} = 3.6\text{pixel} \tag{8}$$

This seems to be a little high. Typical tracking errors are 0.2 pixel to 1 pixel. When the image material is very blurry the tracking error can also be 2 pixel, so even for blurry images the point should be considered to be finite or turned off.